Reports of the Department of Geodetic Science

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SOME PROBLEMS CONCERNED WITH THE GEODETIC USE OF HIGH PRECISION ALTIMETER DATA

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National Aeronautics and Space Administration

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Foreword

This report was prepared by Dr. D. Lelgemann, Visiting Research Associate, Department of Geodetic Science, The Ohio State University, and Wissenschaftl. Rat at the Institut für Angewandte Geodäsie, Federal Republic of Germany. This work was supported, in part, through NASA Grant NGR 36-008-161, The Ohio State University Research Foundation Project No. 3210, which is under the direction of Professor Richard H. Rapp. The grant supporting this research is administered through the Goddard Space Flight Center, Greenbelt, Maryland with Mr. James Marsh as Technical Officer.

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Abstract

The definition of the geoid in view of different height systems is discussed. A definition is suggested which makes it possible to take the influence of the unknown corrections to the various height systems on the solution of Stokes' problem into account.

A solution of Stokes' problem with an accuracy of 10 cm is derived which allows the inclusion of the results of satellite geodesy in an easy way. In addition, equations are developed that may be used to determine spherical harmonics using altimeter measurements, considering the influence of the ellipticity of the reference surface.

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1. Introduction

As part of its Earth and Ocean Physics Application Program (EOPAP) the National Aeronautics and Space Administration (NASA) plans in the next ten years the launch of some satellites equipped with altimeter for ranging to the ocean surface. The announced accuracy of the future altimeter systems lies in the scope of 10 cm.

By solving the inverse problem of Stokes, it is possible to compute gravity anomalies from these very accurate altimeter data. In view of an examination about possibilities, problems and accuracy of a solution of the inverse Stokes' problem with this high accuracy we will treat two preparatory problems concerned with the direct solution.

In order to transform the altimeter data into geoidal undulations (if possible, taking oceanographic informations about the so-called sea surface topography into account) we need a suited definition of the geoid at least with the same accuracy. Another problem is concerned with the impossibility of the measurement of reasonable altimeter data on the continents. So we have to cut the distant zones off in our integral solutions, taking into account their influences by a set of harmonic functions. This is also in agreement with the recent numerical treatment of Stokes formula (Vincent and Marsh, 1973, Rapp, 1973). The best set of harmonic coefficients is, of course, a combination solution. So we should have regard to the fact, that these coefficients do not belong to the potential on the ellipsoid or the earth surface but to a sphere.

The first comprehensive study of the direct solution of Stokes' problem with regard to the use of altimeter data is due to Mather (Mather 1973, 1974). However, because of the necessity of the inclusion of satellite coefficients into the solution, we will follow another way which seems better suited in the case of our preconditions.

The following treatment is based on the results described in (Moritz, 1974). In order to avoid long-winded repetitions we refer often to results and formulae of this study, so that the knowledge of this report may be recommended for an entire insight into the present work.

Regarding our two special problems described above we first try to give a suited definition of the geoid. Of course, we shall not change the definition of the geoid as an equipotential surface of the earth. What we will do is

nothing else than a specialisation of one equipotential surface distinguished as the geoid. Our main condition in this context is the possibility of a realisation of this special equipotential surface by geodetic measurements.

A certain modification of Moritz's approach seems to be necessary if we want to include satellite data (e.g. in form of a set of harmonic coefficients). This is not so in view of the correction terms to Stokes' formula. But the inclusion of satellite information directly in "Stokes approximation" (Moritz 1974, f. (1-9)), may lead to very complicated problems.

In order of a better understanding of our problems and also the way which is chosen for the solution we will remember some basic considerations of geodesy. The main task of geodesy is the estimation of the figure of the earth and the outer gravity field with the aid of suited measurements. In the overwhelming cases these measurements belong to the earth's surface.

We will assume that the earth's surface is a star-shaped surface. In this case any ray from the origin (the gravity center of the earth) intersects this surface only once. Describing the physical surface of the earth by geodetic coordinates, the ellipsoidal height can be considered as a function of the two other coordinates

$$h = h(\varphi, \lambda)$$

h........ellipsoidal height
φ........geodetic latitude
λ........geodetic longitude

We will assume further that we have measurements of the following type

a) ellipsoidal heights h (e.g. by altimetry over the ocean surfaces)

b) potential differences $C = W - W_0$ (by levelling)

c) gravity g

Now by a combination of various types of these data we can obviously solve our main task in different ways. Because of the superficial similarity with the well-known boundary value problems of potential theory we can also consider three geodetic boundary value problems:

a) First geodetic boundary value problem: Given the ellipsoidal height h and the potential difference C. Today, this method looks somewhat artificial, but with the recent development of doppler measurement methods or perhaps with altimetry on the continents, it may become very interesting.

b) Second geodetic boundary value problem: Given the ellipsoidal height h and gravity g. The existence and uniqueness of the solution is discussed by (Koch and Pope, 1972).

c) Third geodetic boundary value problem: Given the potential difference C and gravity g. This is the well-known Molodenskii problem. For a detailed discussion of a solution see (Meissl, 1971).

Supporting on this classification we will now make a few general comments, including a summary of some results.

In all three problems we need a potential value W_0 as additional information. It may be pointed out that by the inclusion of one additional piece of data, (that is in case three, the inclusion of one geometric distance e.g. one ellipsoidal height h) the value W_0 can be computed.

It is nowadays impossible to measure C on the ocean surface. So the determination of the sea surface topography with the aid of geodetic measurements can only be obtained by the solution of problem two (Moritz, 1974). In the case of the inverse problem we must assume that the altimeter information can be corrected by oceanographic information for sea surface topography. In this case the corrected altimeter data should belong to an equipotential surface. Because of this assumption the information $C = 0$ is given in addition to the ellipsoidal height. If the equipotential surface is identical with the geoid, the ellipsoidal heights are identical with the geoidal undulations.

The problem of the unknown geoid and the estimation of datum parameter to the various height systems can be solved by combining data of all three

types. This is what we are going to do in the next two sections. The geoid as the equipotential surface with the value W_G is defined in such a way, that the square sum of the differences to the main height systems (corrected by oceanographic information about sea surface topography) is a minimum. To solve the problem of the practical determination of W_G and to compute the datum corrections to the height systems, condition equations of a least square adjustment procedure are derived in section three. It may be pointed out, that the "geoidal undulations", which are needed in this model as measurements, are not the true geoidal undulations but values obtained from gravity anomalies which are falsified by an unknown correction to the height datums.

The main task of modern geodesy is not the solution of one of the three boundary value problems but a uniform solution which combines data of all types. From a practical point of view the combination of "terrestrial data" (gravity anomalies, altimeter data) and "satellite data" (orbital analysis, satellite to satellite tracking, etc.) is the most important problem. At least the lower harmonic coefficients will be computed from a combination of all these data. From this point of view it is uncomfortable to use solutions of a boundary problem, because the surface of the earth is very complicated. So it is important that the analytical continuation of the potential inside the earth is possible with any wanted degree of accuracy (Krarup, 1969). Moreover, if we start from the same data set (i.e. gravity anomalies) a series evaluation leads to the same formulae as the Molodenskii series solution, as shown by Moritz (Moritz, 1971).

A computational procedure in which we can include terrestrial and satellite data is the following successive reduction method.

A) Direct effect of atmospheric gravity reduction. Remove the atmosphere outside the surface of the earth and redistribute it inside. The resulting disturbing potential is then an analytic function outside of the earth's surface and the reduced gravity anomalies Δg_B are boundary values at the earth's surface.

B) Direct effect of the regard of topography. Compute gravity anomalies Δg_E at the geoid (or the ellipsoid) by a suited form of analytical downward continuation.

C) Direct effect of ellipticity correction. Compute from the gravity anomalies Δg_E at the ellipsoid gravity anomalies Δg_a at the sphere with the radius a.

At this state we can combine the gravity anomalies Δg_a with satellite derived data. From the combined data we can compute the disturbing potential T_a at points on this sphere. We may remember that this potential must not be the true value of the disturbing potential at this point in space but only the result of the analytical continuation (consider the case of a mountain at the equator).

D) Indirect effect of ellipticity correction. Compute the potential T_E at the ellipsoid from the potential values T_a at the sphere with radius a.

E) Indirect effect of the regard of topography. Compute from the potential values T_E at the ellipsoid the potential T_S at the earth's surface by an upward continuation, using the inverse method of step B.

F) Indirect effect of atmospheric gravity reduction. Correct the value T_S at the earth surface by the indirect effect of the atmospheric reduction made in step A.

Here, we will make only some remarks about this method and the results. A detailed description together with a compilation of the formulae is given in section four.

The estimation of the direct and indirect effect of atmospheric gravity reduction is the same as in (Moritz, 1974). The treatment of the influence of the topography is also very similar as used by Moritz. It can be shown (Moritz, 1971) that the common handling of the direct and indirect effect leads to the same formulae as recommended in (Moritz, 1974, sec 4). However, the meaning of the procedure is quite different from Molodenskii's solution, which avoids analytical continuation. But the computational formulae are the same and well suited for practical computations.

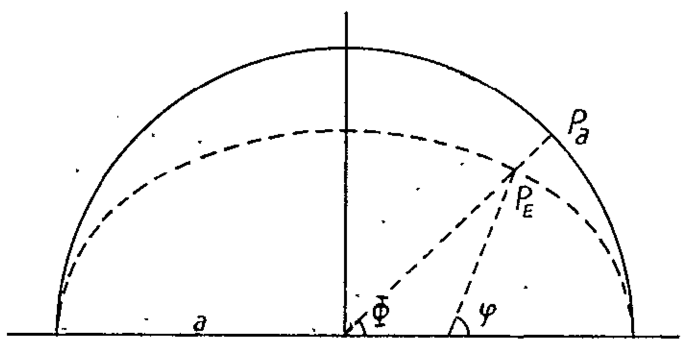
The treatment of the influence of the ellipsoidal shape of the reference surface is different from the procedure in (Moritz, 1974). The first part, the computation of gravity anomalies Δg_a on the sphere with radius a from gravity anomalies Δg_E on the ellipsoid was done in the main in (Lelgemann, 1972). The indirect effect, the computation of T_E at the ellipsoid from T_a at the sphere, is derived in this report.

The use of the final formula for the correction term may also be favorable within the computation of spherical harmonics from given altimeter data. Let us assume that we have altimeter data, corrected for sea surface topography, as a function of the geodetic coordinates. Then we obtain the disturbing potential at sea level by

$$T_E(\varphi, \lambda) = N/\gamma.$$

N.........geoidal undulation (from altimetry)
γ.........normal gravity
T_E........disturbing potential at the geoid or ellipsoid.

In order to obtain a set of spherical harmonics we need the disturbing potential at the sphere with radius a.



We get this value by the simple correction

$$T_a(\Phi, \lambda) = T_E(\varphi, \lambda) - \frac{e'^2}{4} \cos^2\theta \cdot T(\varphi, \lambda)$$

e'.........second eccentricity

The derivation of this simple formula is done in an indirect way. Because of the length of the derivation it is given at the end of the report in the sections 5 to 6.

2. Considerations on the Definition of the Geoid

The first definition of a geoid as that real equipotential surface of the earth gravity potential, which is characterized by the ideal surface of the oceans, was given by Gauss. Such a definition presupposed that the ideal surface of the oceans is part of an equipotential surface of the earth gravitational field. To a certain degree of approximation this idealized sea surface coincides with another more or less time invariant conception, the mean sea level.

We will consider here as mean sea level the mean ocean surface after removing time dependent effects. Because the mean sea level is than not necessarily an equipotential surface of the earth gravitational field, slopes of mean sea level were detected both by levelling and by oceanographic computations. The definition of an ideal surface of the oceans and the computation of the difference between this ideal surface and mean sea level cannot be a problem of geodetic but of oceanographic science.

From a geodetic point of view the idea of a geoid is closely concerned with the definition of the heights. It is well-known that the heights are computed from measured potential differences. Let us assume for the moment the (of course unrealistic) possibility, that we can carry out spirit levelling also over the ocean surfaces. In this case the geodetic community would certainly define as a geoid that equipotential surface, from which the potential differences are counted.

In every case the geoid must be considered as the reference surface of a world wide height system. So within the problem of the definition of a geoid the problem of the definition and also the practical possibilities of the computation of height datums to the various height systems play a central role.

There is a third utilization of the geoid or in this case rather the quasigeoid as established by Molodenskii, (Molodenskii, et. al 1962), which is very important in geodesy and this is the role of the geoidal undulations in the interplay of gravimetric and geometric geodesy. Apart from its own importance, we will use this connection to overcome the problem of the impossibility of spirit levelling over the ocean surfaces.

The most important geodetic aspect in the considerations about a definition of the geoid seems to be the definition of a reference surface for the height determination. So we will mention some principles which should be important in respect of our opinion that the computation of datum corrections is one of our

main problems. As in the case of the definition of other coordinate systems (e.g. the definition of a highly accurate cartesian coordinate system for the purpose of the description of time dependent coordinates of geophysical stations) we may answer the following questions:

1) What physical meaning has the definition?

2) Can we transform the physical definition into a mathematical description?

3) Can we realize the mathematical and physical definition in the real world by measurements?

4) Can we compute transformation parameters to already existing height systems?

We will give the answers in the course of this section with exception of question three, which will be answered in the next section.

After this preliminary considerations we will start the discussion with the definition of a geoid given by (Rapp, 1974). He started the discussion from the set of all equipotential surfaces of the actual gravity field

$$W = W(x, y, z) = \text{const.}$$

W is defined as the sum of the gravity potential W_g and the potential of the atmosphere W_a. We must point out that the potential

$$W = W_g + W_a \tag{2-1}$$

is not harmonic outside of the earth's surface because of the presence of the atmosphere.

Because we are going to distinguish in the following considerations

several different equipotential surfaces, we will call an equipotential surface with the potential W_i (where the subscript i described only the fact, that W_i has a fixed value)

$$\text{geop}\ (W_i).$$

Later on we will specialize one of this equipotential surfaces as the geoid, that is

$$\text{geoid} \doteq \text{geop}\ (W_G).$$

Departing from the customary expression for the potential of the geoid by W_0, we have characterized the potential of the geoid by W_G. The reason for the change of this abbreviation will be clearer in the course of this section. The choice of this special equipotential surface seems in a certain way arbitrarily, dependent on the starting point of the considerations. For this reason we shall discuss, for the moment, the problem separately from the three special areas we have mentioned at the beginning of this section. Then we will look for a combination of all these considerations. Most important, of course, is the possibility of a practical realisation of the geoid in the case when we have enough accurate measuring data.

1) The geop (W_{MSL}) as the ideal surface of the oceans.

(MSL .... mean sea level)

The definition of mean sea level and the ideal surface of the oceans, which we will consider as an equipotential surface, cannot be the task of geodesy but of oceanography. A good description of the difficulties of the definition and more over the realisation of these concepts are given in (Wemelsfelder, 1970).

The following very simple model of the real processes may be sufficient for our considerations. Because geodesy is only interested in the deviation of the ocean surface from an equipotential surface, we may say, that the ideal surface of the ocean is disturbed by the following irregularities:

a) very short periodic irregularities (e.g. ocean waves, swell)

b) periodic or quasi periodic irregularities (e.g. tides)

c) quasistationery irregularities, which retain their form but change their place (e.g. gulf stream)

d) quasistationery irregularities, which retain form and place

If we correct the real ocean surface for all these irregularities, than the result should be an equipotential surface and we will name it by

$$\text{geop}\ (W_{MSL})$$

Corrections of the individual height datums to a world wide height system are then given by the correction (dW_{MSL}) for quasistationery sea surface topography at the water gauges.

Such a definition is not only important for oceanographers, but also for geodesists. If we can compute with the help of oceanographic information the deviation of the sea surface from an equipotential surface, then we can also compute geoidal undulations from altimeter measurements. Especially if we want to recover gravity anomalies from altimeter data we have to use such information.

2) The geop (W_{HSO}) as the basis of a worldwide height system

(HSO............height system zero order)

In order of an explanation of a geop (W_{HSO}) let us start from a reference surface of a particular height system geop (W_{HS1}) (e.g. from the mean sea level 1966.9 at Portland, Maine). We will assume errorless levellings to the reference points of (n-1) additional height systems. Consequently, we have n different height systems with the reference points on equipotential surfaces

$$\text{geop}\ (W_{HSi})$$

Of course, such kind of levelling is impossible because various height systems lay on various continents. We shall bridge this difficulty using the connection between gravimetric and geometric geodesy.

The most plausible reference surface of a worldwide height system is then the equipotential surface geop (W_{HSO}) for which the sum of the square deviations to the particular height systems is a minimum,

$$\sum_{i=1}^{n} (W_{HSO} - W_{HSi})^2 = \text{Min.} \tag{2-2}$$

In this case all height systems have equal influence. As a solution of the problem we get easily

$$W_{HSO} = \frac{1}{n} \sum_{i=1}^{n} W_{HSi} \tag{2-3}$$

We can assume that a geop (W_{HSO}) defined in such a way lies very near the geop (W_{MSL}), because all height systems are based on mean sea levels at least in the reference points. The transformation parameters are given by the definition equation.

3) The geop (W_0) from the connection between gravimetric and geometric geodesy.

($W_0 = U_0 \ldots\ldots U_0$ = normal potential on the surface of the normal ellipsoid)

We will start the definition of a geop (W_0) from the normal potential based on a rotational ellipsoid. The surface of the ellipsoid should be an equipotential surface of the normal potential. It is well known that in this case the normal potential on and outside of the ellipsoid and also the geometric form of the ellipsoid itself can be described by four parameters, e.g.

kM .... mass of the earth
ω .... rotational velocity
J_2 .... harmonic coefficient of order two

U_0.......potential on the ellipsoid surface

or

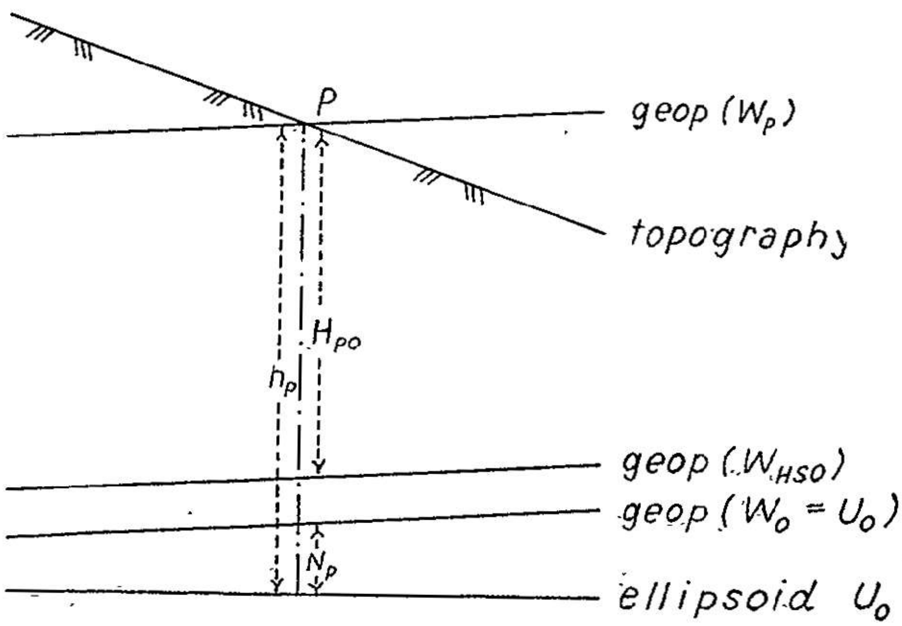
a.........semi major axis

We presuppose that we have very exact values of the first three terms (maybe from satellite geodesy). To a certain degree of accuracy the following relations hold (Heiskanen-Moritz, 1967) for the fourth term

$$U_0 \doteq \frac{kM}{a} \tag{2-4}$$

$$dU_0 \doteq -\frac{kM}{a^2}\,da \doteq -\gamma_0\,da. \tag{2-5}$$

It is well known that we cannot measure the absolute value of the potential. We get this value by an indirect method using e.g. the connection between gravity and a distance in formula (2-5). For a definition of the normal gravity field, it is important that only three physical constants are fixed values of the real earth. The fourth term is in certain limits arbitrarily. Let us describe the geometrical relationship in this case. We have the following situation,



h_P.........ellipsoidal height
N_P ....geoidal undulation
H_{PO}........orthometric height of the point P in a worldwide height system

It is easily seen that in the case outlined above, the main equation,

$$h = H + N, \tag{2-6}$$

connecting gravimetric and geometric geodesy holds not in this form. We have two possibilities to correct the situation. We can refer the heights to the reference surface

$$\text{geop } (W_O)$$

or we can change the size of the normal ellipsoid by

$$d U_O = (W_{HSO} - W_O).$$

The choice of the kind of the correction is our own pleasure. On the other hand if we have fixed one of both values (that is W_{HSO} or W_O, either by a mark on the earth surface or by a given number), the difference between them must be computed from geodetic measurements.

In this connection we will also consider the mathematical description of our problem. For this purpose we use the relation

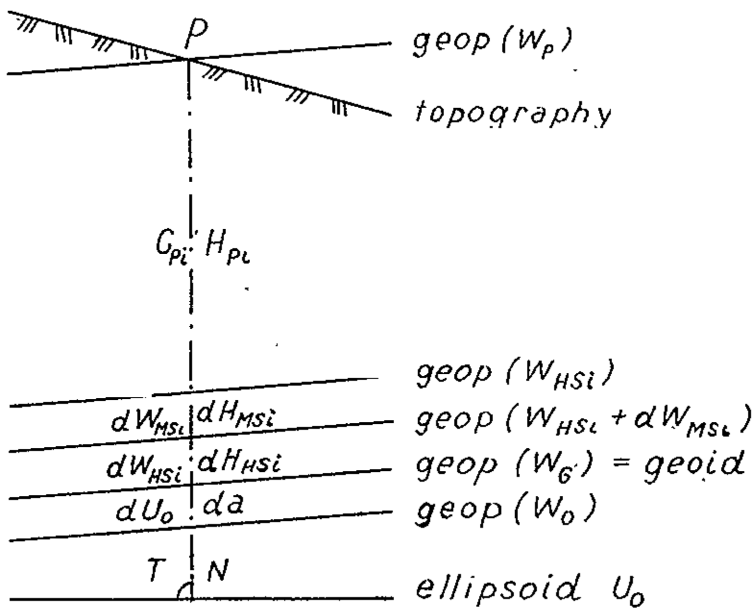
$$h = H + N = H^* + \zeta \tag{2-7}$$

h.........ellipsoidal height
H ....orthometric height
N ....geoidal undulation
ζ.......quasi-geoidal undulation
H^*.......normal height

For an accurate definition and explanation of all these terms see e.g. (Heiskanen-Moritz, 1967).

4) The geoid = geop (W_G) as a result of the previous considerations

Because of the close connection with a worldwide height system our final definition of the geoid is based mainly on point two of the previous considerations. In addition we will take into account information from oceanographic science. In order to bridge the continents and to get a numerical value of W_G we shall use also the connection between gravimetric and geometric geodesy. We study the geometrical relationship on the following figure.



The geoid is defined by

(2-8) $$\sum_{i=1}^{n} \left(W_G - (W_{HSi} + dW_{MSi}) \right)^2 = \min.$$

H_{Pi} .... orthometric (or normal) height in a particular height system i

dH_{MSi}......... reduction of the gauge (mean sea

level) mark to an ideal ocean surface because of sea surface topography, given from oceanographic science

dH_{HSi} .... deviation of the corrected basic level of the height system i from the geoid, defined by (2-8) and constant for the area of this particular height system

da .... correction of the semi-major axis for the term $(W_O - W_G)$

N .... geoidal undulation

In an explicit form we have for the definition of the geoid

$$W_G = \frac{1}{n} \sum_{i=1}^{n} (W_{HSi} + dW_{MSi}). \tag{2-9}$$

If the oceanographic corrections dW_{MSi} are correct then all geop $(W_{HSi} + dW_{MSi})$ are identical or at least close together and the corrections dW_{HSi} are small or zero. In this case the practical procedure developed in the next section may be regarded as an independent checking of the oceanographic information by geodetic methods. If we have no oceanographic information we can put, in this case, simply $dW_{MSi} = 0$.

3. On the Realisation of the Definition of the Geoid

In the previous section the possibility of transferring the theoretical definition into physical reality and vice versa was one of our main requirements. This is certainly a question which can only be answered by statistical methods, that is by the development of a suited adjustment model.

Our mathematical description of the problem is based on the connection between gravimetric and geometric geodesy. So we can start with the condition that the basic equation of gravimetric geodesy

$$h = H + N = \zeta + H^*$$

is fulfilled in a set of m points. Because the observations from levelling, from geometric and from gravimetric geodesy may be given in different systems we must include in our model transformation parameters as unknowns. In this way we are lead to the model of a least square adjustment of condition equations with additional unknowns

$$A^T v + B^T x + w = 0. \tag{3-1}$$

The solution of such a system is well known (e.g. Gotthardt, 1968, p. 238 ff). We will discuss here the explicit form of the condition equations, presupposing that the following "observations" are given at m points P_j on the earth surface:

h_j .... ellipsoidal height, computed from rectangular coordinates as a result of satellite geodesy

H^*_{ij} .... normal height in the i-th system of n height systems

ζ_j .... quasi-geoidal undulation

In addition, we have for any of the n height systems a constant dH_{MSi}, representing sea surface topography as computed in oceanographic science:

dH_{MSi} .... gauge correction due to sea surface topography in the i-th height system

In order to fulfill the condition equation

$$h - H^* - \zeta = 0 \tag{3-2}$$

we will first consider our "measurements" (h_j, H^*_{ij}, ζ_j).

As mentioned above these "observations" may belong to different systems. The transformation parameters between these systems may not be known and have to be estimated in the course of the adjustment. This is true in any case for the height datums dH_{HSi} and the correction of the semi-major axis da.

In this way it is possible to take also other systematic effects into account. We will restrict ourselves to the unknown parameters described in the following context.

<u>a) Ellipsoidal Height h</u>

We assume that the geometric reference ellipsoid is not in an absolute position, but connected with the center of mass by the vector (dx_0, dy_0, dz_0). Then for absolute ellipsoidal heights h we get the following equation (Heiskanen-Moritz, 1967, p. 207)

$$h = h_j - \cos\varphi\cos\lambda\, dx_0 - \cos\varphi\sin\lambda\, dy_0 - \sin\varphi\, dz_0 \doteq da. \tag{3-3}$$

We have already included in this equation the unknown

$$da \doteq -\frac{1}{\gamma}\, dU_0 \tag{3-4}$$

from which we can compute the numerical value of the potential at the geoid by

$$W_G = U_0 + dU_0 \tag{3-5}$$

after the adjustment.

b) Normal Height H^*

From figure two we get easily

$$H^* = H_{ij}^* + dH_{MSi} + dH_{HSi} \tag{3-6}$$

The n unknowns dH_{HSi} are the transformation parameters for all the height systems together with the constant values dH_{MSi}.

c) Quasi-geoidal undulations ζ

This case is less trivial, because the "measurements" ζ_j are connected with the unknowns dH_{HSi} by Stokes' integral formula. With regard to our adjustment model we must derive a linear relationship between the "measurement" ζ_j and its true value ζ

$$\zeta = \zeta_j + \sum_{k=1}^{n} (a_{kj} \cdot dH_{HSk}) \tag{3-7}$$

The influence of ζ_j by the unknowns dH_{HSk} is due to the fact that we can compute gravity anomalies Δg only with heights related to the geop $(W_{HSk} + dW_{MSk})$. For gravity anomalies Δg_G related to the geoid we have the expression

$$\Delta g_G = g - \gamma_E + 0.3086\,[(H_k + dH_{MSk}) + dH_{HSk}] \tag{3-8}$$

or

$$\Delta g_G = \Delta g_k + 0.3086\, dH_{HSk} \tag{3-9}$$

with

$$\Delta g_k = (g - \gamma_E + 0.3086\,(H_k + dH_{MSk})\ ).$$

We have used in this formulae the normal gradient of gravity

$$F \doteq 0.3086\,h$$

obtaining F in mgal if we introduce h in meters. The accuracy of the formulae above is sufficient for the computation of the coefficients a_{kj}.

The use of Δg_k is in agreement with our conception, that the first step to the estimation of the geoid is the inclusion of oceanographic information. The gravity anomalies Δg_k are related to the geop $(W_{HSk} + dW_{MSk})$, which should be very closed to the geoid or, in an ideal case, should already coincide with the geoid. If we put the expression (3-9) into Stokes' formula, we get

$$\zeta = \zeta_1 + f\zeta$$

where

$$f\zeta = \frac{R}{4\pi G} \int_{\sigma} (0.3086\ dH_{HSk})\, S(\psi)\, d\sigma. \tag{3-10}$$

In spherical approximation we can write

$$0.3086 \doteq -\frac{\partial \gamma}{\partial h} \doteq \frac{2G}{R}\ .$$

Putting this relation into (3-10), we obtain

$$(3\text{-}11) \qquad f\zeta = \frac{1}{2\pi} \int_{\sigma} dH_{HSk} \cdot S(\psi)\, d\sigma.$$

The area in which heights related to the geop (W_{HSk}) are being used, may be described by F_{HSk}.

Because dH_{HSk} is constant over the area F_{HSk}, we get the following expression

$$(3\text{-}12) \qquad f\zeta = \sum_{k=1}^{n} dH_{HSk} \cdot \frac{1}{2\pi} \int_{F_{HSk}} S(\psi)\, d\sigma$$

or

$$(3\text{-}13) \qquad a_{kj} = \frac{1}{2\pi} \int_{F_{HSk}} S(\psi_j)\, d\sigma$$

where $S(\psi_j)$ is Stokes' function referred to the point P_j in question. From the expression (3-13) we are able to compute numerical values for all the coefficients a_{kj}, replacing the integral by a summation.

In order to get a feeling about the magnitude of a_{kj} we will consider a simple example. We assume only one height system with a reference surface different from the geoid. We assume further that the area F_{HS1} of this height system is a spherical cap of size ψ_0 around the point P_j under consideration. Starting with

$$a_{kj} = \frac{1}{2\pi} \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} S(\psi) \sin\psi \, d\psi \, d\alpha$$

the integration over α gives

$$a_{kj} = \int_{\psi=0}^{\psi_o} S(\psi) \sin\psi \, d\psi = 2\,J(\psi_o)$$

where the function $J(\psi)$ is defined by (Heiskanen-Moritz, 1967, p. 119)

$$J(\psi) = \tfrac{1}{2} \int_0^{\psi} S(\psi) \sin\psi \, d\psi .$$

From a table of $J(\psi)$ (e.g. Lambert and Darling, 1936) we draw the values

$$J(\psi) = 0.5 \quad \text{at} \quad \psi \doteq 27^\circ \quad \text{and} \quad \psi = 50^\circ .$$

Hence, under this circumstances the coefficient of the unknown dH_{HSj} is

$$(1 + a_{jj}) \doteq 2.$$

After this preliminary discussion about the connection of the "measurements" and the unknown parameters, we will now derive the explicit form of the condition equations. We put (3-3), (3-6) and (3-7) into (3-2)

$$(h_j + V_{1j}) - (H^*_{1j} + V_{2j}) - (\zeta_j + V_{3j}) - \cos\varphi\cos\lambda\, dx_o$$

$$\text{(3-14)} \qquad - \cos\varphi\sin\lambda\, dy_o - \sin\varphi\, dz_o - da - dH_{MSj} - dH_{HSj}$$

$$- \sum_{k=1}^{n} (a_{kj} \cdot dH_{HSk}) = 0.$$

To this system of condition equations, we have to add the equation

$$\sum_{i=1}^{n} dH_{HSi} = 0 \tag{3-15}$$

which is a consequence of the definition equation (2-8). This additional condition equation connects only unknowns and not the "observations". However, there are no principal problems concerned with the solution if we add equation (3-15) at the end of the equation system (Gotthardt, 1967).

We will now discuss some main aspects related to the present and future accuracy of the data. We seperate the discussion into four parts in accordance with the different type of data which are needed in the adjustment.

a) Determination of gauge corrections dH_{MSi}

For purposes of physical oceanography to interpret the results of a satellite altimeter, it is necessary to know the absolute shape of a level surface near mean sea level. Also for geodetic purposes it is necessary that we know the deviation of mean sea level from an equipotential surface. We can compute this deviation combining altimeter measurements and gravity measurements (Moritz, 1974, sec. 5). However, gravity measurements over sea are very time-consuming. So from a practical point of view it could be very helpful if we can determine geoidal undulations directly from altimeter measurements taking a small correction term from oceanographic science into account.

To compare the results from oceanographic science with the geodetic results it is of course very important to determine the heights of several gauges in the same height system. Differences between theory and measurements are known (AGU, 1974). The first step in the realisation of a worldwide geoid should be an explanation of these differences, because this seems to be an individual problem which can perhaps be solved prior to the inclusion of geometric results and geoidal computations. From a comparison of levelling and the results of oceanographic science at the coasts of the United States (AGU, 1974) we can conclude that the present accuracy of the oceanographic computations of sea surface topography is better than a meter.

b) Determination of station coordinates and the ellipsoidal height h_i

We expect accurate station coordinates from satellite geodesy, deriving ellipsoidal heights from the cartesian coordinates. The present accuracy lies in the order of 5-10 m (Mueller, 1974). However, with the high accuracy of new developed laser systems (some cm) and with special satellites such as Lageos or special methods, such as lunar laser ranging we can expect a fast increase of the accuracy at least for the coordinates of some geophysical stations. So it seems advantageous to use such geophysical stations also as the basis for the adjustment model developed above. It should be useful to have several well distributed stations in the area F_{HSi} of every main height system. Necessary are likewise the very accurate parameter of a normal ellipsoid, that is at least the three parameters (ω, J_2, kM), computing the semi-major axis a within the definition of the geoid.

c) Determination of normal heights H^* or orthometric heights H

For long times levelling was one of the most accurate geodetic measurments with a standard error up to ±0.1 mm per km distance. Today, nevertheless, the increasing accuracy of distance measurements let us expect a similar accuracy for coordinates and distances. To stay comparable in the accuracy a very careful examination of systematical errors in levelling is required.

So far it is possible, a mutual connection of the geophysical stations and also the connection of these stations and the fundamental gauges of the height system by high precision levellings should be performed.

d) Determination of geoidal heights N or quasi-geoidal undulations ζ

This seems from the present accuracy considerations the most crucial point in the method. On the other hand the definition of the geoid given above is connected with precise gravimetric geodesy, so that we can assume the presence of data with the necessary accuracy.

This is not the case for present gravity data which allows a computation of geoidal undulations with an accuracy of better than 10 m. Methods in recent development like aero gradiometry and so on will provide perhaps a much better estimation.

However, a highly accurate determination of the harmonic coefficients of lower order and a good and dense gravity material around the geophysical stations seems of high value in the solution of our problem.

4. On the Solution of Stokes' Problem Including Satellite Data Information

We will not consider Stokes' problem as the computation of a regularized geoid but as the following problem: It may be possible to determine the disturbing potential on and outside of the earth, using gravity data from the earth's surface in Stokes' integral formula and correct the result by some small terms. In this way we can interpret Molodenskii's solution as a well suited and theoretical unobjectionable solution of Stokes' problem.

On the other hand, we have to go nowadays a step further. Our data set comprise not exclusively gravity measurements on the surface of the earth but also a lot of other information about the gravity potential, one of the most important the lower harmonic coefficients from satellite geodesy.

We do not know an exact solution of Stokes' problem. We will discuss only approximate solutions, but the approximation error must be less than 10 cm in the geoidal undulations and the solution should be as simple as possible in view of practical computations.

Because of the presence of the atmosphere, we have to solve not a Laplace but a Poisson equation. We shall overcome this difficulty by a suitable gravity reduction.

The majority of the data (the gravity anomalies) calls for a solution of the so-called Molodenskii problem. This is a very complicated type of a non-linear boundary value problem. Molodenskii already has based his solution on Stokes' formula, because the result of Stokes' formula differs from the exact solution only by small correction terms.

A unified treatment of Stokes' problem with the necessary accuracy of better than 10 cm for geoidal undulations was done by Moritz (Moritz, 1974). Such a solution must be taken into account,

1) the effect of the atmosphere
2) the influence of topography,
3) the ellipticity of the reference surface.

Neglecting terms of higher order of a Taylor series expansion, Moritz treats all three effects independently of each other. He ends with the following procedure:

1) Reduce Δg to the "Stokes' approximation" Δg^o by,

$$\Delta g^o = \Delta g - \sum_{i=1}^{3} G_i. \tag{4-1}$$

2) Apply Stokes' integral to Δg_o

$$\zeta^o = \frac{R}{4\pi\gamma^o} \iint_\sigma \Delta g^o \, S(\psi)\, d\sigma. \tag{4-2}$$

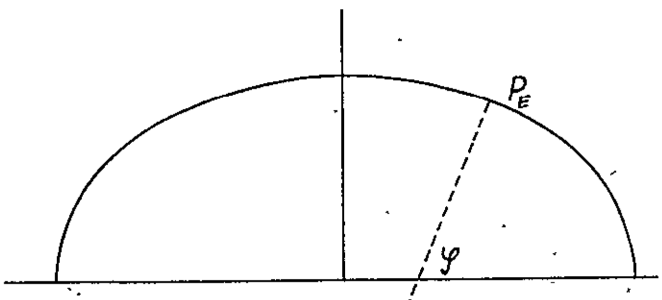
3) Correct ζ^o to obtain the actual value ζ by,

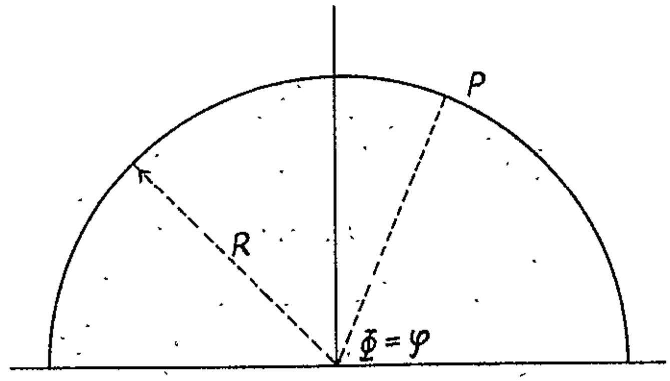
$$\zeta = \zeta^o + \sum_{i=1}^{3} z_i. \tag{4-3}$$

This solution of Stokes' problem is well suited if only gravity anomalies are at hand. However, the inclusion of data other than gravity anomalies may improve the solution very much. At present, the most important additional data are, without question, the results of satellite geodesy in the form of potential coefficients. It seems very difficult to include this data in formula (4-2) in a convenient way.

For this reason we have to change the model of Moritz. For a better unde standing of the basic idea, we will explain the difference in the case of the application of the ellipsoidal corrections.

In (Moritz, 1974) and also in (Lelgemann, 1970), we have established a one to one mapping of the reference ellipsoid on a sphere with radius R by mapping a point P of geodetic (geographical) coordinates (φ, λ) on the ellipsoid into a point P' of spherical coordinates $(\Phi = \varphi, \lambda)$ on the sphere.





In this case the sphere is not attached to the ellipsoid, serving only as an auxiliary surface for the computations and the result belongs, of course, to the point P on the ellipsoid (or rather to the earth's surface).

Now, we will describe the present model. In this case, we compute from the anomalies on the ellipsoid the anomalies on the sphere with radius a, which is tangent to the ellipsoid at the equator. With the aid of Stokes' integral, we compute T_a at the point P' on the sphere (if P' lies outside of the earth's surface, T_a is the real disturbing potential at the point P' in space). At this step, we can combine the surface data with satellite data. Finally, we compute from the disturbing potential at the sphere with radius a the disturbing potential T_E at the surface of the ellipsoid. This treatment of the ellipticity seems to be the appropriate expansion of the correction for spherical approximation in view of the fact, that the analytical continuation of the disturbing potential is possible with any wanted accuracy.

In order to get a closed theory, we must also treat the influence of topography in another way than Molodenskii. Molodenskii's solution is identical with the analytical continuation to point level (Moritz, 1971). The formulae for analytical continuation are derived by Moritz in the cited publication. Because there is no theoretical difference in the reduction to different level surfaces (Moritz,

1971, sec. 10) we can first reduce the measurements to sea level and afterwards back to the earth's surface. The formulae remain nearly the same, but we should look at the terms of the present model as a result of a computation of the potential of analytical continuation at the ellipsoid and the sphere with radius a. In this way we end with the following procedure:

1) Reduce the gravity anomalies for the effect of the atmospher~

(4-4) $$\Delta g_s = \Delta g - \delta g_1 .$$

2) Compute by a suited form of analytical downward continuation gravity anomalies at the geoid or better, immediately at the ellipsoid

(4-5) $$\Delta g_E = \Delta g_s - \delta g_2 .$$

3) Compute from the gravity anomalies on the ellipsoid the gravity anomalies on the sphere with radius a

(4-6) $$\Delta g_a = \Delta g_E - \delta g_3 .$$

4) Apply Stokes' integral in order to get the potential (of analytical continuation) at the sphere with radius a

(4-7) $$T_a = \frac{a}{4\pi} \iint_\sigma (\Delta g_a)\, S(\psi)\, d\sigma .$$

5) Compute the potential at the ellipsoid by

(4-8) $$T_E = T_a + \delta t_3 .$$

6) Compute the disturbing potential at the earth's surface by upward continuation

(4-9) $$T_s = T_E + \delta t_2$$

7) Correct the disturbing potential at the earth's surface by the indirect effect of removing the atmosphere

(4-10) $$T = T_s + \delta t_1 .$$

From the disturbing potential T we get immediately the quasi geoidal undulations by

(4-11) $$\zeta = \frac{T}{\gamma}$$

and the geoidal undulations by

(4-12) $$N = \zeta + (H^* - H).$$

Now, we will collect the formulae for the correction terms, together with the references. For a detailed explanation of these formulae see the references.

(4-13) 1) $$\delta g_1 = - \frac{k M_A(r)}{r^2} = - \delta g_A .$$

(See Moritz, 1974, formula (2-23)). Note that this correction has the same absolute value as the gravity correction δg_A in (IAG, 1971, page 72). It takes its maximum value of $|\delta g_A| = 0.87$ mgal at sea level.

(4-14) 2) $$\delta g_2 = H \cdot L_1(\Delta g)$$

with

(4-15) $$L_1(\Delta g) = \frac{R^2}{2\pi} \iint_\sigma \frac{\Delta g - \Delta g_P}{\ell_0^3} \, d\sigma .$$

(See Moritz, 1971, formulae (1-5), (1-8)). Note tha this is only the first term of a series solution. However, it seems to be sufficient for all practical purposes, especially in the case of the use of altimeter data).

(4-16) 3) $$\delta g_3 = -\frac{e'^2}{a}\sum_{n=2}^{\infty}\sum_{m=0}^{n}(n-1)\left[C_{nm}R_{nm}(\theta,\lambda)+D_{nm}S_{nm}(\theta,\lambda)\right]$$

The coefficients can be computed in the following manner. When

(4-17) $$T(\theta,\lambda) = \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left[A_{nm}R_{nm}(\theta,\lambda)+B_{nm}S_{nm}(\theta,\lambda)\right]$$

then

(4-18)
$$C_{nm} = A_{(n-2)m}P_{nm} + A_{nm}q_{nm} + A_{(n+2)m}r_{nm}$$
$$D_{nm} = B_{(n-2)m}P_{nm} + B_{nm}q_{nm} + B_{(n+2)m}r_{nm}$$

where,

(4-19)
$$P_{nm} = \frac{(5n-17)(n-m-1)(n-m)}{4(n-1)(2n-3)(2n-1)}$$
$$q_{nm} = \frac{-6n^3+8n^2+25n+6nm^2+6m^2+21}{4(n-1)(2n+3)(2n-1)}$$
$$r_{nm} = \frac{(5n+11)(n+m+2)(n+m+1)}{4(n-1)(2n+5)(2n+3)}$$

See (6-33), (6-35) and (6-41) and also (Lelgemann, 1972, formula (33)). The formulae give the reduction term in

the case of gravity anomalies. Similar formulae for the computation of gravity disturbances at the sphere from gravity disturbances at the ellipsoid are derived in section 6 of this study.

4) $$T_a = \frac{a}{4\pi} \iint_\sigma (\Delta g_a)\, S(\psi)\, d\sigma.$$

The disturbing potential T is computed instead of geoidal undulations in order to avoid the definition of geoidal undulations or quasi geoidal undulations in space.

(4-20) 5) $$\delta t_3 = \frac{e'^2}{4} \cdot \cos^2\theta \cdot T(\theta, \lambda).$$

The derivation of this simple formula is rather lengthy. It is given in the last three sections of this study, (see formula (6-35)).

(4-21) 6) $$\delta t_2 = - H \cdot \Delta g$$

(See Moritz, 1971, formula (1-14)). Note that this correction is almost zero in the case of altimeter data because of $H \doteq 0$.

(4-22) 7) $$\delta t_1 = - \int_r^\infty \frac{k\, M(r')}{r'^2}\, dr'$$

or

(4-23) $$\delta t_1 = \int_r^\infty \delta g_1(r)\, dr'$$

(See Moritz, 1974, formulae (2-22) and (2-27). Note

that the indirect effect amounts to maximal 0.6 cm.
In spite of the overall accuracy of the solution, we
can neglect its influence.

All the correction formulae are the same in the case of gravity disturbances with the exception of the correction term δg_3. The correction term $\delta\overline{g}_3$ for gravity disturbances is given by formula (6-33).

5. The Influence of the Ellipticity in the Case of Gravity Disturbances.

The purpose of the next three sections is a unified treatment of the influence of the ellipsoidal reference surface and an unified evaluation of the formulae which connect the potential on the ellipsoid and the potential in space (that is on the sphere with radius a). The content of the present section should only be seen as an intermediate result, which will be needed in the following section.

In contrast to the very similar derivations in (Lelgemann, 1970) the derivation of the whole theory is based on gravity disturbances. The advantages rest on the avoiding of the difficulties concerned with the spherical harmonics of zero and the first order, which appeared in (Lelgemann, 1970).

We will mention that the first explicit solution based on gravity disturbances as data was derived by Moritz (Moritz, 1974), starting from the solution of the problem for gravity anomalies. Using his technique in an inverse way we shall derive in the next section the solution for gravity anomalies from the solution based on gravity disturbances.

On the surface of the normal ellipsoid we have in a linear approximation the following boundary condition,

(5-1)
$$\left(\frac{\partial T}{\partial n}\right)_E = -\delta g .$$

Using Green's second formula for the function T and the ellipsoid as the integration surface,

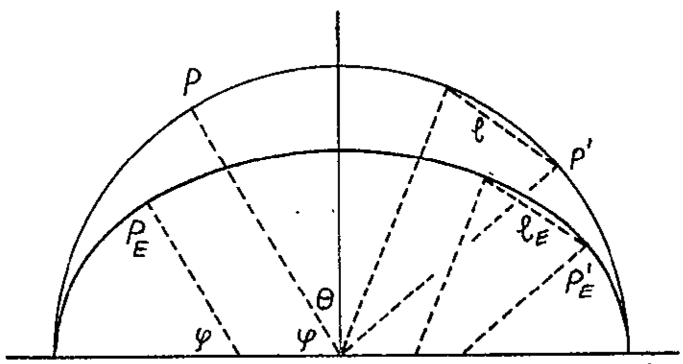
(5-2)
$$-2\pi T = \int_E \left(\frac{1}{\ell_E}\frac{\partial T}{\partial n} - T\frac{\partial}{\partial n}\left(\frac{1}{\ell_E}\right)\right) dE$$

we get, after inserting the boundary condition, the integral equation

(5-3)
$$2\pi T = \int_E \frac{\delta g}{\ell_E}\, dE + \int_E T\left(\frac{\partial}{\partial n}\left(\frac{1}{\ell_E}\right)\right) dE.$$

In consequence of the assumption above, this is a Fredholm integral equation of the second kind. The integrals must be taken over the ellipsoid. In order to get a solution we transform first the integrals into a spherical coordinate system. The result is a linear integral equation with an unsymmetric kernel. We shall develop this kernel with the required accuracy into a power series of e'^2. The resulting system of integral equations consists only of equations with symmetric kernels, moreover of equations with a well known kernel. Because the eigen-functions of the integral equations are the spherical harmonics we arrive very easy to series solutions.

We map points of the ellipsoid with the geodetic latitude φ in such a manner onto the sphere that the spherical latitude is identical with the geodetic latitude (left side of the figure)



$\theta = 90° - \varphi$

First we shall evaluate some terms in the powers of e'^2 up to the required degree of accuracy. For the surface element of the ellipsoid we have,

(5-4) $$M = a(1+e'^2)^{1/2}(1+e'^2\cos^2\varphi)^{-3/2} \doteq a(1-e'^2 + 3/2\, e'^2\cos^2\theta)$$

(5-5) $$N = a(1+e'^2)^{1/2}(1+e'^2\cos^2\varphi)^{-1/2} \doteq a(1+\tfrac{1}{2}e'^2\cos^2\theta)$$

(5-6) $$dE = M \cdot N \cdot \cos\varphi\, d\varphi\, d\lambda = M \cdot N \cdot d\sigma \doteq (1-e'^2 + 2e'^2\cos^2\theta)\, a^2 d\sigma.$$

Further we will represent $1/\ell_E$ in dependence of $1/\ell$ (see figure above). We start with

$$(5\text{-}7) \qquad \ell_E^2 = (X_E - X_E')^2 + (Y_E - Y_E')^2 + (Z_E - Z_E')^2$$

where,

$$X_E = N\cos\varphi\cos\lambda, \quad Y_E = N\cos\varphi\sin\lambda, \quad Z_E = b^2/a^2\, N\sin\varphi.$$

Developing this formula in powers of e'^2 and using the substitution.

$$(5\text{-}8) \qquad \ell^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2$$

with

$$X = a\cos\varphi\cos\lambda, \quad Y = a\cos\varphi\sin\lambda, \quad Z = a\sin\varphi$$

we get

$$(5\text{-}9) \qquad \ell_E^2 \doteq \ell^2 + \tfrac{1}{2}e'^2(\cos^2\theta - \cos^2\theta')\,\ell^2 - 2e'^2a^2(\cos\theta - \cos\theta')^2$$

and with the help of a series evaluation the final result,

$$(5\text{-}10) \qquad \frac{1}{\ell_E} \doteq \frac{1}{\ell}\left[1 - \frac{1}{4}e'^2(\cos^2\theta + \cos^2\theta') + e'^2a^2\frac{(\cos\theta - \cos\theta')^2}{\ell^2}\right]$$

An exact expression for $\frac{\partial}{\partial_n}(1/\ell_E)$ was derived by Molodenskii (Molodenskii, 1961, page 54)

(5-11) $$\frac{\partial}{\partial n}\left(\frac{1}{\ell_E}\right) = \frac{1}{2\ell_E N}(1 + e'^2 n^2)$$

with

(5-12) $$-n = \frac{1}{\ell_E}\frac{b^2}{a^2}(N\sin\varphi - N'\sin\varphi').$$

A series evaluation gives

(5-13) $$\frac{\partial}{\partial n}\left(\frac{1}{\ell_E}\right) \doteq -\frac{1}{2a\ell_E}\left(1 - \tfrac{1}{2}e'^2\cos^2\theta + e'^2\frac{(\cos\theta - \cos\theta')^2}{\ell_O^2}\right)$$

with

(5-14) $$\ell_O = 1/a \cdot \ell = 2\sin\psi/2.$$

Inserting these expressions into the integral equation (5-3), we obtain, after some transformations

$$2\pi T \doteq \int_\sigma \left\{1 - e'^2 + \frac{7}{4}e'^2\cos^2\theta - \frac{1}{4}e'^2\cos^2\theta' + e'^2\frac{(\cos\theta - \cos\theta')^2}{\ell_O^2}\right\}$$

(5-15) $$\cdot\ \delta g \cdot \frac{a}{\ell_O}d\sigma + \int_\sigma\left\{-\frac{1}{2} + \frac{e'^2}{2} - \frac{5}{8}e'^2\cos^2\theta + \frac{1}{8}e'^2\cos^2\theta'\right.$$

$$\left. - e'^2\frac{(\cos\theta - \cos\theta')^2}{\ell_O^2}\right\} \cdot T \cdot \frac{1}{\ell_O}d\sigma.$$

The kernel of this integral equation is not symmetric. So we evaluate T in powers of the small term e'^2

(5-16) $$T = T^0 + e'^2 T^1 + e'^4 T^2 + \ldots..$$

Inserting this series into the integral equation (5-15) and equating the coefficients of the powers of e'^2 we arrived at the following integral equation system

$$2\pi T^0 = \int_\sigma a \cdot \delta g \cdot \frac{1}{\ell_0}\, d\sigma - \frac{1}{2}\int_\sigma T^0 \frac{1}{\ell_0}\, d\sigma$$

$$2\pi T^1 = \int_\sigma a \cdot \delta g \cdot \frac{1}{\ell_0}\Big[-1 + \frac{7}{4}\cos^2\theta - \frac{1}{4}\cos^2\theta'$$

(5-17) $$+ \frac{(\cos\theta - \cos\theta')^2}{\ell_0^2}\Big]\, d\sigma + \int_\sigma T^0 \frac{1}{\ell_0}\Big[\frac{1}{2} - \frac{5}{8}\cos^2\theta$$

$$+ \frac{1}{8}\cos^2\theta' - \frac{(\cos\theta - \cos\theta')^2}{\ell_0^2}\Big]\, d\sigma - \frac{1}{2}\int T^1 \frac{1}{\ell_0}\, d\sigma.$$

$$2\pi T^2 = \ldots\ldots\ldots$$

we must solve successively the first and second equation of this system. These two equations contain all terms up to the order e'^2 so that a solution of the further equations is not necessary in view of the required accuracy.

We develop the first integral equation into a series of eigenfunctions. Because,

(5-18) $$\frac{1}{4\pi}\int_\sigma X_n(\theta, \lambda)\; P_n(\cos\psi)\, d\sigma = \frac{1}{2n+1}\; X_n(\theta', \lambda')$$

we get the series representation,

(5-19) $$T = a \sum_{n=0}^{\infty} \frac{\delta g_n}{n+1}\,.$$

Analogous to Stokes formula we can represent this series as an integral formula (Hotine, 1969)

$$T = \frac{a}{4\pi} \int_\sigma \delta g \; \tilde{S}(\psi) \, d\sigma \tag{5-20}$$

with

$$\tilde{S}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \tag{5-21}$$

or by the closed expression

$$\tilde{S}(\psi) = \frac{1}{\sin\psi/2} - \ln\left(1 + \frac{1}{\sin\psi/2}\right). \tag{5-22}$$

In order to get a solution of the second equation of the system (5-17), we evaluate the disturbing term in a series of harmonic functions. Because T^1 is to be multiplied with e'^2 we can use the following formulae in spherical approximation:

$$\delta g = \frac{1}{a} \sum_{n=0}^{\infty} (n+1)\, T_n \tag{5-23}$$

and

$$T = \sum_{n=0}^{\infty} T_n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{nm} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{A_{nm} R_{nm}(\theta, \lambda) + B_{nm} S_{nm}(\theta, \lambda)\}. \tag{5-24}$$

As an intermediate result we get

(5-25)
$$T^1 + \frac{1}{4\pi}\int_\sigma T^1 \frac{d\sigma}{\ell_0} = A_1 + A_2 + A_3 + A_4$$

with

(5-26)
$$A_1 = -\frac{1}{4\pi}\int_\sigma \left[\sum_{n=0}(2n+1)\ T_n\right]\frac{d\sigma}{\ell_0} = -\sum_{n=0} T_n$$

(5-27)
$$= -\frac{\cos^2\theta}{4\cdot 4\pi}\int_\sigma \left[\sum_{n=0}^{\infty}(2n+1)\ T_n\right]\frac{d\sigma}{\ell_0} = -\frac{\cos^2\theta}{4}\sum_{n=0}^{\infty} T_n.$$

Using formula (8-6) we obtain

(5-28)
$$A_3 = \frac{1}{4\pi}\int_\sigma \frac{(\cos\theta - \cos\theta')^2}{\ell_0^3}\sum_{n=0}^{\infty} 2\cdot n\cdot T_n\, d\sigma =$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2n(4m^2-1)}{(2n+3)(2n-1)(2n+1)}\ T_{nm}$$

and with formula (8-5)

$$A_4 = \frac{1}{4\pi}\int_\sigma \left[\sum_{n=0}^{\infty}\frac{(14n+9)}{4}\ T_n\right]\cos^2\theta\ \frac{d\sigma}{\ell_0}$$

(5-29)
$$\frac{1}{4}\sum_{n=0}^{\infty}(14n+9)\sum_{m=0}^{n}\left[A_{nm}\left\{\frac{\alpha_{nm}}{2n+5}\ R_{(n+2)m}(\theta,\lambda)\right.\right.$$
$$\left. + \frac{\beta_{nm}}{2n+1}\ R_{nm}(\theta,\lambda) + \frac{\gamma_{nm}}{2n-3}\ R_{(n-2)m}(\theta,\lambda)\right\} +$$

$$+ B_{nm} \cdot \left\{ \frac{\alpha_{nm}}{2n+5} S_{(n+2)m}(\theta, \lambda) + \frac{\beta_{nm}}{2n+1} S_{nm}(\theta, \lambda) + \frac{\gamma_{nm}}{2n-3} S_{(n-2)m}(\theta, \lambda) \right\} \Bigg] .$$

Performing the integrations on the left hand side of the integral equation we have

$$T^1 + \frac{1}{4\pi} \int_\sigma T^1 \frac{d\sigma}{\ell_0} = 2 \sum_{n=0}^{\infty} \frac{n+1}{2n+1} T_n^1 . \tag{5-30}$$

Transforming first the term A_2 in a series of harmonics we get afterwards from a series comparison on both sides of the integral equation the following solution,

$$T^1 = -\tfrac{1}{2} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} T_n + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{n(4m^2 - 1)}{(2n+3)(2n-1)(n+1)} T_{nm}$$

$$+ \tfrac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \Bigg[A_{nm} \left\{ \frac{3n+1}{n+3} \alpha_{nm} R_{(n+2)m}(\theta, \lambda) + \frac{3n+2}{n+1} \beta_{nm} R_{nm}(\theta, \lambda) \right.$$

(5-31)

$$\left. + \frac{3n+3}{n-1} \gamma_{nm} R_{(n-2)m}(\theta, \lambda) \right\} + B_{nm} \left\{ \frac{3n+1}{n+3} \alpha_{nm} S_{(n+2)m}(\theta, \lambda) \right.$$

$$\left. + \frac{3n+2}{n+1} \beta_{nm} S_{nm}(\theta, \lambda) + \frac{3n+3}{n-1} \gamma_{nm} S_{(n-2)m}(\theta, \lambda) \right\} \Bigg] .$$

It is possible, of course, to write this result in a more convenient form. However, we are going to use this result only in order to derive the relationship between the disturbing potential on the sphere with radius a from the disturbing potential on the ellipsoid, which is done in the following section. The expression (5-31) is very suited for this purpose.

6. The Connection Between the Potential on the Ellipsoid and the Potential in Space

In the previous section we have mapped data from the ellipsoid to an auxiliary sphere, solved an integral equation for the desired function and remapped the solution to the ellipsoid. In this section we will compute the potential on the tangent sphere of radius a as a function of boundary values on the ellipsoid.

We can write the potential (of analytical continuation) on the sphere with radius a as a series of harmonics

$$T_a = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[A^a_{nm} R_{nm}(\Theta, \lambda) + B^a_{nm} S_{nm}(\Theta, \lambda) \right] \tag{6-1}$$

Θ ....complement of the geocentric latitude Φ

$\left.\begin{matrix} R_{nm}(\Theta, \lambda) \\ S_{nm}(\Theta, \lambda) \end{matrix}\right\}$ unnormalized spherical harmonics in view of the simpler recursion formulae. They have the same definition as in (Heiskanen-Moritz, 1967).

As data on the surface of the ellipsoid we shall consider δg and later on also T and Δg.

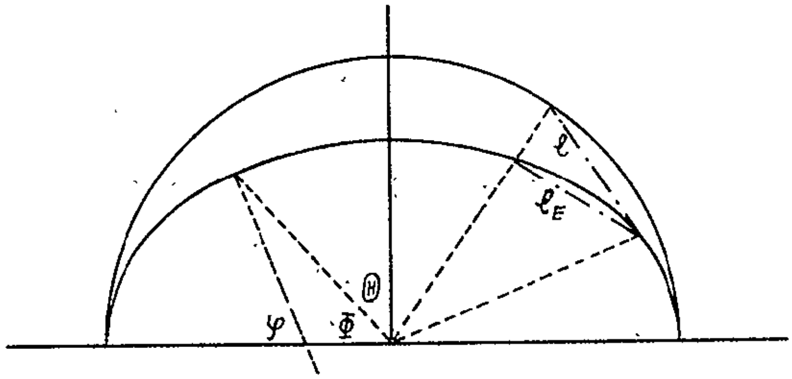
We start the derivation with the representation of the disturbing potential outside the ellipsoid by the well-known formula of a surface layer χ

$$T_a = \int_E \frac{\chi}{\ell} \, dE \tag{6-2}$$

together with the integral equation for the surface layer

$$2\pi\chi - \frac{\partial}{\partial n} \int_E \frac{\chi}{\ell_E} \, dE = \delta g \tag{6-3}$$

The definition of the terms, especially of ℓ and ℓ_E can be seen from the figure,



n .... direction of the outer ellipsoid normal

Θ .... $= 90 - \Phi$

Similar as in the previous section we must transform the integral equation for the surface layer χ into an equation over the unit sphere, using geocentric latitudes Φ as parameters.

For this purpose, we can use again the expression

$$(6\text{-}4) \qquad \frac{\partial}{\partial n}\left(\frac{1}{\ell_E}\right) \doteq -\frac{1}{2a\ell_E}\left(1 - \tfrac{1}{2}e'^2\cos^2\Theta' + e'^2\,\frac{(\cos\Theta - \cos\Theta')^2}{\ell_0^2}\right)$$

taking now the derivative in the fixed point P'.

Together with (Molodenskii 1961, page 56)

$$(6\text{-}5) \qquad \frac{dE}{\ell_E} \doteq \frac{a}{\ell_0}\,(1 - \tfrac{3}{4}e'^2\sin^2\Phi)(1 + \tfrac{1}{4}e'^2\sin^2\Phi')\cos\Phi\, d\Phi\, d\lambda$$

we arrive with the transformation onto the sphere at

$$(6\text{-}6) \qquad 2\pi\chi + \tfrac{1}{2}\int_\sigma \chi \frac{d\sigma}{\ell_0} + \frac{e'^2}{2}\int_\sigma \left[-\frac{3}{4}\cos^2\Theta - \frac{1}{4}\cos^2\Theta' + \frac{(\cos\Theta - \cos\Theta')^2}{r_0^2}\right]\chi \frac{d\sigma}{\ell_0} = \delta g.$$

Now we evaluate the surface layer χ in powers of e'^2

$$\chi = \chi^0 + e'^2\chi^1 + e'^4\chi^2 + \dots\dots\dots\dots$$

Substituting this expression in the integral equation (6-6) and equating the coefficients of the same power of e'^2 we arrive at the following system of equations

$$(6\text{-}7) \qquad 2\pi\chi^0 + \tfrac{1}{2}\int_\sigma \chi^0 \frac{d\sigma}{\ell_0} = \delta g$$

$$\chi^1 + \frac{1}{4\pi}\int_\sigma \chi^1 \frac{d\sigma}{\ell_0} = + \frac{1}{4\pi}\int_\sigma \left[\frac{3}{4}\cos^2\Theta + \frac{1}{4}\cos^2\Theta' - \frac{(\cos\Theta - \cos\Theta')^2}{\ell_0^2}\right]\chi^0 \frac{d\sigma}{\ell_0}$$

$$\chi^2 = \dots\dots\dots\dots\dots\dots$$

Again, we are only interested in the first and second equation. With formula (5-18) we have also due to the orthogonality relations of spherical harmonics

$$(6\text{-}8) \qquad \frac{1}{4\pi}\int_\sigma X_n(\Theta, \lambda)\frac{d\sigma}{\ell_0} = \frac{1}{2n+1} X_n(\Theta', \lambda')$$

and with this the solution of the first equation

$$\chi^{0} = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \delta g_n . \tag{6-9}$$

In order to solve the second equation we must expand the right hand side in a series of harmonics. As χ^1 has still to be multiplied by e'^2 we can use the formulae in spherical approximation

$$\delta g_n = \frac{n+1}{a} T_n \tag{6-10}$$

and also

$$\chi_n = \frac{1}{4\pi a} (2n+1) T_n \tag{6-11}$$

with

$$T_n = \sum_{n=0}^{n} [A_{nm} R_{nm}(\Theta, \lambda) + B_{nm} S_{nm}(\Theta, \lambda)] .$$

Then we get

$$\sum_{n=0}^{\infty} \frac{2n+2}{2n+1} \chi_n^1 = \frac{1}{4\pi a} \left\{ \frac{\cos^2 \Theta'}{4} \sum_{n=0}^{\infty} T_n \right.$$

$$- \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(4m^2 - 1)}{(2n+3)(2n-1)} T_{nm}$$

$$+ \frac{3}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{nm} \left[\frac{(2n+1)}{(2n+5)} \alpha_{nm} R_{(n+2)m}(\Theta, \lambda) \right.$$

$$\left. + \beta_{nm} R_{nm}(\Theta, \lambda) + \frac{(2n+1)}{(2n-3)} \gamma_{nm} R_{(n-2)m}(\Theta, \lambda) \right] \Bigg\}$$

Developing also $\cos^2\Theta' T_n$ with the help of formula (8-5) and comparing the two sides of the equation results in

$$\chi^1 = \frac{1}{8\pi a} \left\{ - \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(2n+1)(4m^2-1)}{(n+1)(2n+3)(2n-1)} T_{nm}(\Theta, \lambda) \right.$$

(6-12)
$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{nm} \left[\frac{2(n+1)}{n+3} \alpha_{nm} R_{(n+2)m}(\Theta, \lambda) + \frac{(2n+1)}{(n+1)} \right.$$

$$\left. \left. \cdot \beta_{nm} R_{nm}(\Theta, \lambda) + \frac{2n}{(n-1)} \gamma_{nm} R_{(n-2)m}(\Theta, \lambda) \right] \right\}$$

The solutions (6-9) and (6-12) must be inserted into (6-2). With (Molodenskii 1962, page 56)

(6-13)
$$dE \doteq a^2 (1 - e'^2 \sin^2 \Phi) \cos \Phi \, d\Phi \, d\lambda$$

(6-14)
$$\ell \doteq 2 \cdot \sin \psi/2 \cdot \sqrt{a \cdot r_E}$$

(6-15)
$$r_E \doteq a (1 - \tfrac{1}{2} e'^2 \sin^2 \Phi)$$

we get

(6-16)
$$\frac{dE}{\ell} \doteq \frac{a}{\ell_0} (1 - \tfrac{3}{4} e'^2 \cos^2 \Theta) \cos \Phi \, d\Phi \, d\lambda$$

and

$$(6\text{-}17) \qquad T_a = \int_E \frac{\chi}{\ell}\, dE = a \int_{\sigma} \chi \left(1 - \frac{3}{4} e'^2 \cos^2\Theta\right) \frac{1}{\ell_0} \cos\Phi \, d\Phi \, d\lambda .$$

Evaluating also T into a power series of the small parameter e'^2

$$T_a = \widetilde{T}_0 + e'^2 \widetilde{T}^1 + e'^4 \widetilde{T}^2 + \cdots\cdots$$

and inserting this expression in (6-17) we obtain

$$(6\text{-}18) \qquad \widetilde{T}^0 = a \int_{\sigma} \chi^0 \frac{d\sigma}{\ell_0}$$

and

$$(6\text{-}19) \qquad \widetilde{T}^1 = -a \int_{\sigma} \frac{3}{4} \cos^2\Theta \, \chi^0 \frac{d\sigma}{\ell_0} + a \int_{\sigma} \chi^1 \frac{d\sigma}{\ell_0}$$

We get as an intermediate solution

$$(6\text{-}20) \qquad \widetilde{T}^0 = a \sum_{n=0}^{\infty} \frac{1}{n+1} \delta g_n$$

or written as an integral equation

$$(6\text{-}21) \qquad 2\pi \widetilde{T}^0 + \frac{1}{2} \int_{\sigma} \widetilde{T}^0 \frac{d\sigma}{\ell_0} = a \int_{\sigma} \delta g \frac{d\sigma}{\ell_0}$$

and afterwards

$$\widetilde{T}^1 = -\tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(4m^2-1)}{(n+1)(2n+3)(2n-1)}\,T_{nm}(\Theta,\lambda)$$

(6-22)
$$-\frac{1}{4}\sum_{n=0}^{\infty}(3n+1)\sum_{m=0}^{n}A_{nm}\left\{\frac{\alpha_{nm}}{(n+3)}R_{(n+2)m}(\Theta,\lambda)\right.$$

$$\left.+\frac{\beta_{nm}}{(n+1)}R_{nm}(\Theta,\lambda)+\frac{\gamma_{nm}}{(n-1)}R_{(n-2)m}(\Theta,\lambda)\right\}$$

Whereas usually the integrations over the unit sphere are made with the aid of the geodetic latitude φ, the geocentric latitudes Φ are used as parameters in equation (6-21). Applying here (Lelgemann, 1970)

(6-23)
$$\cos\Phi \doteq (1+e'^2\sin^2\varphi)\cos\varphi$$

(6-24)
$$d\Phi \doteq (1-e'^2+2e'^2\sin^2\varphi)\,d\varphi$$

(6-25)
$$\frac{1}{\ell_0} = \frac{1}{2\sin\bar{\Psi}/2} \doteq \frac{1}{2\sin\psi/2}\left[1-\frac{e'^2}{2}\right.$$

$$\left.\cdot(\cos^2\theta+\cos^2\theta')+e'^2\frac{(\cos\theta-\cos\theta')^2}{4\sin^2\psi/2}\right]$$

with

$$\theta = 90-\varphi$$

$$\cos\bar{\Psi} = \cos\Phi\cos\Phi' + \sin\Phi\sin\Phi'\cos\Delta\lambda$$

$$\cos\psi = \cos\varphi\cos\varphi + \sin\varphi\sin\varphi\cos\Delta\lambda$$

we arrive at the following integral equation system

$$T^0 + \frac{1}{4\pi} \int_\sigma T^0 \frac{d\sigma}{\ell_0} = \frac{a}{2\pi} \int_\sigma \delta g \frac{d\sigma}{\ell_0}$$

$$T^1 + \frac{1}{4\pi} \int_\sigma T^1 \frac{d\sigma}{\ell_0} = - \frac{1}{4\pi} \int_\sigma T^0 \left\{ - 1 - \tfrac{1}{2}\cos^2\theta' + \tfrac{5}{2}\cos^2\theta \right.$$

(6-26)
$$\left. + \frac{(\cos\theta - \cos\theta')^2}{4\sin^2\psi/2} \right\} \frac{d\sigma}{\ell_0} + \frac{a}{2\pi} \int_\sigma \delta g \left\{ -1 - \tfrac{1}{2}\cos^2\theta' \right.$$

$$\left. + \tfrac{5}{2}\cos^2\theta + \frac{(\cos\theta - \cos\theta')^2}{4\sin^2\psi/2} \right\} \frac{d\sigma}{\ell_0}$$

$$T^2 = \cdots\cdots\cdots\cdots\cdots\cdots$$

with

$$d\sigma = \sin\theta \, d\theta \, d\lambda.$$

We have in spherical approximation

$$2a\,\delta g = \sum_{n=0}^{\infty} (2n+2)\,T_n$$

and therefore

(6-27)
$$T^1 + \frac{1}{4\pi} \int_\sigma T^1 \frac{d\sigma}{\ell_0} = \frac{1}{4\pi} \int_\sigma \left\{ \sum_{n=0}^{\infty} (2n+1)\,T_n \right\} \left\{ -1 \right.$$

$$- \tfrac{1}{2}\cos^2\theta' + \tfrac{5}{2}\cos^2\theta + \frac{(\cos\theta - \cos\theta')^2}{4\sin^2\psi/2} \Bigg\} \frac{d\sigma}{\ell_0} .$$

A series evaluation gives

$$T^1 = -\tfrac{1}{2}\sum_{n=0}^{\infty} \frac{(2n+1)}{(n+1)} T_n + \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(2n+1)(4m^2-1)}{(n+1)(2n+3)(2n-1)} T_{nm}$$

(6-28)
$$+ \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm} \Bigg\{ \frac{2n}{n+3}\,\alpha_{nm} R_{(n+2)m}(\theta,\lambda) + \frac{2n+1}{n+1}\,\beta_{nm} R_{nm}(\theta,\lambda)$$

$$+ \frac{2n+2}{n-1}\,\gamma_{nm} R_{(n-2)m}(\theta,\lambda) \Bigg\}$$

Adding now (6-22) and (6-28) we get

$$T^1 = -\tfrac{1}{2}\sum_{n=0}^{\infty} \frac{2n+1}{n+1} T_n + \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{n(4m^2-1)}{(n+1)(2n+3)(2n-1)} T_{nm}$$

(6-29)
$$+ \frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm} \Bigg\{ \frac{5n-1}{n+3}\,\alpha_{nm} R_{(n+2)m}(\theta,\lambda)$$

$$+ \frac{5n+3}{n+1}\,\beta_{nm} R_{nm}(\theta,\lambda) + \frac{5n+7}{n-1}\,\gamma_{nm} R_{(n-2)m}(\theta,\lambda) \Bigg\}.$$

We sum up the result by

$$T^1 = \frac{1}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{ A_{nm} [\bar{p}_{nm} R_{(n+2)m}(\theta, \lambda) + \bar{q}_{nm} R_{nm}(\theta, \lambda)$$

$$+ \bar{r}_{nm} R_{(n-2)m}(\theta, \lambda)] + B_{nm} [\bar{p}_{nm} S_{(n+2)m}(\theta, \lambda) + \bar{q}_{nm} S_{nm}(\theta, \lambda)$$

$$+ \bar{r}_{nm} S_{(n-2)m}(\theta, \lambda)] \}$$

with

$$\bar{p}_{nm} = \frac{(5n-1)(n-m+1)(n-m+2)}{(n+3)(2n+1)(2n+3)}$$

$$\bar{q}_{nm} = \frac{-6n^3 - 8n^2 + 6nm^2 - 6m^2 + n + 3}{(n+1)(2n+3)(2n-1)}$$

$$\bar{r}_{nm} = \frac{(5n+7)(n+m)(n+m-1)}{(n-1)(2n+1)(2n-1)} \quad .$$

In the case of the disturbing potential of the earth we have

$$A_{00} = A_{10} = A_{11} = B_{11} = 0 .$$

We get a proper expression of the final result from the solution of (6-21) together with a simple renumbering of the terms of this series:

(6-30) $$T_a = T^0 + e'^2 T^1$$

(6-31) $$T^0(\theta, \lambda) = a \sum_{n=2}^{\infty} \frac{1}{n+1} \delta g_n(\theta, \lambda)$$

$$T^1 = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[C^a_{nm} R_{nm}(\theta, \lambda) + D^a_{nm} S_{nm}(\theta, \lambda) \right] \tag{6-32}$$

$$C^a_{nm} = \left[A_{(n-2)m} \cdot p_{nm} + A_{nm} q_{nm} + A_{(n-2)m} r_{nm} \right]$$

$$D^a_{nm} = \left[B_{(n-2)m} p_{nm} + B_{nm} q_{nm} + B_{(n-2)m} r_{nm} \right]$$

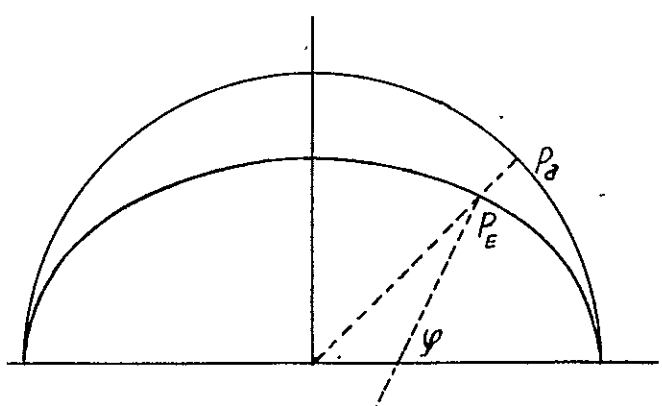
with

$$p_{nm} = \frac{(5n - 11)(n - m - 1)(n - m)}{4(n + 1)(2n - 3)(2n - 1)}$$

$$q_{nm} = \frac{-6n^3 - 8n^2 + 6nm^2 - 6m^2 + n + 3}{4(n + 1)(2n - 1)(2n + 3)}$$

$$r_{nm} = \frac{(5n + 17)(n + m + 2)(n + m + 1)}{4(n + 1)(2n + 5)(2n + 3)}$$

It may be pointed out that the gravity disturbances belong to points $P_E(\varphi, \lambda)$ on the ellipsoid and the disturbing potential to points $P_a(\Phi, \lambda)$ on the sphere.



Based on this result, we will derive some other useful formulae. We begin with a direct relation between the gravity disturbances at the ellipsoid δg_E and the gravity disturbances δg_a at the tangent sphere

(6-33) $$\delta g_a = \delta g_E - \delta\overline{g}_3 .$$

We use the abbreviation ($\delta\overline{g}_3$) because the abbreviation (δg_3) is already reserved for the gravity anomalies (see formula (4-6)). The potential T_a is given on a sphere. So we have the exact series relation

(6-34) $$\delta g_a = \frac{1}{a} \sum_{n=0}^{\infty} (n+1)(T_a)_n .$$

Inserting (6-30) into (6-34) we get as the desired result

(6-35) $$\delta g_a = \delta g_E + \frac{e'^2}{a} \sum_{n=0}^{\infty} (n+1) \sum_{m=0}^{n} [\, C^a_{nm} R_{nm}(\theta, \lambda) + D^a_{nm} S_{nm}(\theta, \lambda) \,] .$$

or

$$\delta\overline{g}_3 = - \frac{e'^2}{a} \sum_{n=0}^{\infty} (n+1) \sum_{m=0}^{n} [\, C^a_{nm} R_{nm}(\theta, \lambda) + D^a_{nm} S_{nm}(\theta, \lambda) \,] .$$

Next, we will derive a direct relation between the disturbing potential T_a at the tangent sphere and the disturbing potential T_E at the ellipsoid. This derivation should be performed in three steps. First we compute δg_a from T_a with the help of (6-20). Second, we compute the gravity disturbances at the ellipsoid with (6-33), using for more convenience the intermediate relation (6-29). Third, we compute the disturbing potential T_E at the ellipsoid with formulae (5-19) and (5-31) of the previous section. We arrive with the simple result

(6-36) $$T_E = T_a + \delta t_3$$

with

$$\delta t_3 = \frac{e'^2}{4} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \{ A_{nm} [\alpha_{nm} R_{(n+2)m}(\theta, \lambda) + \beta_{nm} R_{nm}(\theta, \lambda)$$

$$+ \gamma_{nm} R_{(n-2)m}(\theta, \lambda)] + B_{nm} [\alpha_{nm} S_{(n+2)m}(\theta, \lambda)$$

$$+ \beta_{nm} R_{nm}(\theta, \lambda) + \gamma_{nm} R_{(n-2)m}(\theta, \lambda)] \} .$$

or with the aid of (8-5)

$$\delta t_3 = \frac{e'^2}{4} \cos^2\theta \; T(\theta, \lambda) . \tag{6-37}$$

Because δt_3 is a term of second order we can neglect the difference between θ and Θ, writing also

$$\delta t_3 = \frac{e'^2}{4} \cos^2\Theta \; T(\Theta, \lambda) .$$

Finally, we shall derive the relation between the gravity anomalies Δg_a at the tangent sphere and the gravity anomalies Δg_E at the ellipsoid, starting from the known relation at the ellipsoid (Lelgemann, 1970)

$$\Delta g_E = \delta g + \frac{1}{\gamma} \frac{\partial r}{\partial n} T \doteq \delta g_E - \frac{2}{a} (1 + e'^2 - e'^2 \cos^2\theta) T_E . \tag{6-38}$$

We have by definition

$$\Delta g_a = \Delta g_E - \delta g_3 \tag{6-39}$$

and

$$\Delta g_a = \delta g_a - \frac{2}{a} T_a . \tag{6-40}$$

After some manipulation we get from these formulae

$$\delta g_3 = \delta \bar{g}_3 - \frac{2}{a}(1 + e'^2 - e'^2 \cos^2 \theta) T_\varepsilon + \frac{2}{a} T_a$$

or as a final result

$$\delta g_3 = \delta \bar{g}_3 - \frac{e'^2}{a}(2 - \tfrac{3}{2} \cos^2 \theta)\; T(\theta, \lambda) . \tag{6-41}$$

It is very easy to derive (4-16), (4-17), (4-18) and (4-19) from (6-33), (6-35) and (6-41).

7. Summary

The definition of the geoid is discussed in the first part of the present study. Of course, the geoid is a certain equipotential surface of the earth gravity field. The discussion is only concerned with a specification of this equipotential surface. The final proposition for a definition is based on the geodetic height systems by an inclusion of given information from oceanographic science about sea surface topography.

The realisation of the geoid should be made in two steps. First, the existing height systems are corrected by information about sea surface topography. An equipotential surface is choosen in such a manner that the sum of the squares of the deviations to the corrected main height systems is a minimum. This particular equipotential surface is called the geoid.

A least squares procedure is derived for a realisation of this definition. The precise cartesian coordinates of geophysical stations and in addition the quasi geoidal undulations and the normal heights in these points are needed as data. The unknown corrections to the various height datums influence, of course the gravity anomalies and consequently the geoidal undulations. This influence is regarded in the least squares procedure.

In the second part of the study a solution of Stokes' problem with an accuracy of better than ±10 cm is suggested, in which the results of satellite geodesy can be included in a rather simple way. The method is based on the possibility that the potential outside the earth surface can be approximated by the potential of analytical continuation inside the earth with any accuracy. Gravity anomalies from the earth's surface are reduced by three successive corrections to gravity anomalies at the sphere with radius a. At this sphere they can be combined with given potential coefficients from satellite geodesy. Then the potential at the surface of the earth is computed with the aid of additional three corrections.

Of special interest could be the very simple formula which connects the potential at the ellipsoid with the potential at the sphere with radius a. In the computation of potential coefficients from altimeter data the influence of ellipticity can easily be taken into account with this formula.

8. Appendix

We shall derive in the appendix an evaluation of the function

$$\frac{(\cos\theta - \cos\theta')^2}{8\sin^3\psi/2} \tag{8-1}$$

θ.... polar distance
ψ.... spherical distance between two points
P' and P

in a series of spherical harmonics, starting with the derivation of two formulae which we shall need in the final evaluation.

As the immediate result of the kernel of Poisson's integral formula (Heiskanen-Moritz, 1967)

$$\frac{R(r^2 - R^2)}{\ell^3} = \sum_{n=0}^{\infty} (2n+1)\left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi)$$

we get for $r = R$

$$\sum_{n=0}^{\infty} (2n+1) P_n(\cos\psi) = 0. \tag{8-2}$$

Now, we will try to derive a series evaluation for

$$\frac{1}{\ell^3} = \frac{1}{8\sin^3\psi/2} \quad .$$

In this case, we do not get a well defined expression due to the relation of (8-2). The evaluation can be started with the well known formula

$$\frac{1}{\ell} = \frac{1}{[(r - R)^2 + 4rR\sin^2\psi/2]^{1/2}} = \sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} P_n(\cos\psi).$$

The first derivative gives

$$\frac{\partial(1/\ell)}{\partial r} = -\sum_{n=0}^{\infty} (n+1) \frac{R^n}{r^{n+2}} P_n(\cos\psi)$$

$$= -\frac{1}{\ell^3} [(r - R) + 2R\sin^2\psi/2].$$

The second derivative gives the result

$$\frac{\partial^2(1/\ell)}{\partial r} = \sum_{n=0}^{\infty} (n+1)(n+2) \frac{R^n}{r^{n+3}} P_n(\cos\psi)$$

$$= -\frac{1}{\ell^3} + 3\frac{1}{\ell^5} [(r - R) + 2R\sin^2\psi/2]^2.$$

On the sphere with the radius R we have

$$\ell_0 = 2R\sin\psi/2$$

For points on this sphere we can write

$$\left[\frac{\partial^2(1/\ell)}{\partial r}\right]_{r=R} = \frac{1}{R^3}\sum_{n=0}^{\infty} (n+1)(n+2) P_n(\cos\psi)$$

$$= -\frac{1}{8R^3\sin^3\psi/2} + \frac{3}{8R^3\sin\psi/2}$$

A further evaluation gives

$$\frac{1}{8\sin^3\psi/2} = \sum_{n=0}^{\infty} (n+1)(n+2)\,P_n(\cos\psi) - \frac{3}{4}\cdot\frac{1}{2\sin\psi/2}.$$

or

$$\frac{1}{8\sin^3\psi/2} = \sum_{n=0}^{\infty} \left[(n+1)(n+2) - \frac{3}{4}\right] P_n(\cos\psi)$$

With the identity

$$(n+1)(n+2) - \frac{3}{4} = \frac{1}{4}(2n+1)(2n+5)$$

we get

$$\frac{1}{8\sin^3\psi/2} = \frac{1}{4}\sum_{n=0}^{\infty} (2n+1)(2n+5)\,P_n(\cos\psi).$$

Because of (8-2) we can see that the evaluation of $1/\ell_Q^3$ into a series of spherical harmonics gives not a well defined result but an expression of the form

(8-3) $$\frac{1}{8\sin^3\psi/2} = \sum_{n=0}^{\infty} \frac{(2n+1)(n+a)}{2}\,P_n(\cos$$

where a is any given number. On the other hand we shall see that in the series expression for formula (8-1) the term a drops out.

Let us start with the following expression for a fixed point P' (Heiskanen-Moritz, 1967).

$$f(\theta, \lambda) = \frac{(\cos\theta - \cos\theta')^2}{8\sin^3\psi/2} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} [\, a_{nm}(\theta', \lambda')\, R_{nm}(\theta, \lambda)$$

$$+ \, b_{nm}(\theta', \lambda')\, S_{nm}(\theta, \lambda)\,].$$

Then we can estimate the coefficients a_{nm} by

$$a_{nm} = -\frac{2n+1}{2\pi}\,\frac{(n-m)\,!}{(n+m)\,!}\int_{\sigma}\{\,[\,(\cos\theta - \cos\theta')^2 \cdot$$

$$\sum_{n=0}^{\infty}\frac{(2n+1)(n+a)}{2}\, P_n(\cos\psi)\,]\, R_{nm}(\theta, \lambda)\,\}\, d\sigma.$$

Using twice the recursion formula for Legendre functions

$$\cos\theta\, P_{nm}(\cos\theta) = \frac{n-m+1}{2n+1}\, R_{(n+1)m}(\cos\theta) + \frac{n+m}{2n+1}\, P_{(n-1)m}(\cos\theta)\,.$$

and multiplying the result with $\cos(m\lambda)$ we get the following two expressions

(8-4) $$\cos\theta\, R_{nm}(\theta, \lambda) = \frac{n-m+1}{2n+1}\, R_{(n+1)m}(\theta, \lambda) + \frac{n+m}{2n+1}\, R_{(n-1)m}(\theta, \lambda)$$

and

(8-5) $$\cos^2\theta\, R_{nm}(\theta, \lambda) = \alpha_{nm} R_{(n+2)m}(\theta, \lambda) + \beta_{nm} R_{nm}(\theta, \lambda) + \gamma_{nm} R_{(n-2)m}(\theta, \lambda)$$

with

$$\alpha_{nm} = \frac{(n - m + 1)(n - m + 2)}{(2n + 1)(2n + 3)}$$

$$\beta_{nm} = \frac{2n^2 - 2m^2 + 2n - 1}{(2n + 3)(2n - 1)}$$

$$\gamma_{nm} = \frac{(n + m)(n + m - 1)}{(2n + 1)(2n - 1)}$$

With the help of this recursion formula we get, due to the orthogonality relations of spherical harmonics, after the integration

$$a_{nm}(\theta', \lambda') = \frac{(n - m)!}{(n + m)!} \Big[-(2n + 1)(n + a + 2)\, \alpha_{nm} R_{(n+2)m}(\theta', \lambda')$$

$$- (2n + 1)(n + a)\, \beta_{nm} R_{nm}(\theta', \lambda')$$

$$- (2n + 1)(n + a - 2)\, \gamma_{nm} R_{(n-2)m}(\theta', \lambda')$$

$$+ 2 \cos\theta' (n - m + 1)(n + a + 1) R_{(n+1)m}(\theta', \lambda')$$

$$+ 2 \cos\theta' (n + m)(n + a - 1) R_{(n-1)m}(\theta', \lambda')$$

$$- \cos^2\theta' (2n + 1)(n + a) R_{nm}(\theta', \lambda') \Big]$$

Using the recursion formula once more the term a drops out and we are left with the result

$$a_{nm} = 2 \frac{(n - m)!}{(n + m)!} \left[\frac{(n - m + 1)(n + m + 1)}{(2n + 3)} - \frac{(n + m)(n - m)}{(2n - 1)} \right]$$

$$\cdot R_{nm}(\theta', \lambda')$$

or

$$a_{nm} = 2 \frac{(n-m)!}{(n+m)!} \quad \frac{4m^2 - 1}{(2n+3)(2n-1)} R_{nm}(\theta', \lambda').$$

Analogous evaluations give

$$b_{nm} = 2 \frac{(n-m)!}{(n+m)!} \quad \frac{4m^2 - 1}{(2n+3)(2n-1)} S_{nm}(\theta', \lambda').$$

As a final result, we get the desired expression

(8-6)
$$\frac{(\cos\theta - \cos\theta')^2}{8\sin^3\psi/2} =$$

$$\sum_{n=0}^{\infty} \left\{ - \frac{1}{(2n+3)(2n-1)} P_n(\cos\theta)\, P_n(\cos\theta') \right.$$

$$+ \quad 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \quad \frac{4m^2 - 1}{(2n+3)(2n-1)} \left[R_{nm}(\theta', \lambda')\, R_{nm}(\theta, \lambda) \right.$$

$$\left. \left. + \quad S_{nm}(\theta', \lambda')\, S_{nm}(\theta, \lambda) \right] \right\}.$$

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